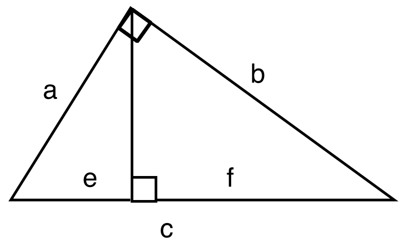
Pythagorean Theorem

There are many ways to prove the Pythagorean Theorem. Here are a few:

Triangle abc is a right triangle and an altitude is drawn, splitting side c into parts e and f.



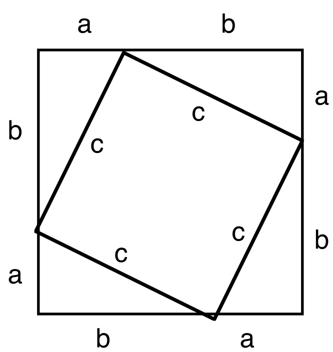
By similar triangles, $\frac{e}{a} = \frac{a}{c}$ and $\frac{f}{b} = \frac{b}{c}$

So that $a^2 = ec$ and $b^2 = fc$

Adding them together, we get

$$a^2 + b^2 = ec + fc$$
$$a^2 + b^2 = (e + f)c$$
$$\therefore a^2 + b^2 = c^2$$

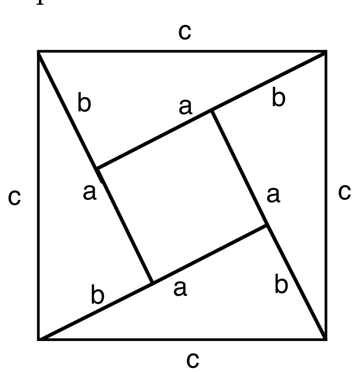
A square with side c is inscribed in another square, breaking the larger sides into a and b.



The area of the big square is equal to the area of the little square plus the area of the four right triangles, so that

$$(a+b)^2 = 4\left(\tfrac{1}{2}ab\right) + c^2$$
$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$\therefore a^2 + b^2 = c^2$$

Square c is divided into four right triangles (abc) and a small square (side = a-b).



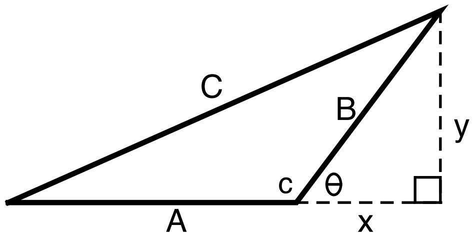
The area of the big square is equal to the area of the little square plus the area of the four right triangles, so that

$$4\left(\tfrac{1}{2}ab\right) + (a-b)^2 = c^2$$
$$2ab + a^2 - 2ab + b^2 = c^2$$
$$\therefore a^2 + b^2 = c^2$$

Pythagorean Theorem

Here are other useful extensions of basic trig stuff that can be useful in physics:

The Law of Cosines



We start of with a random triangle ABC as shown in bold to the left. Then we drop an altitude as shown by the dashed line *y*, making a right triangle with C as the hypoteneuse, and also a right triangle with B as the hypoteneuse.

We can use the Pythagorean Theorem on both right triangles:

$$C^2 = (A+x)^2 + y^2 = A^2 + x^2 + y^2 + 2Ax$$
$$B^2 = x^2 + y^2$$

We can then substitute the second into the first to get

$$C^2 = A^2 + B^2 + 2Ax$$

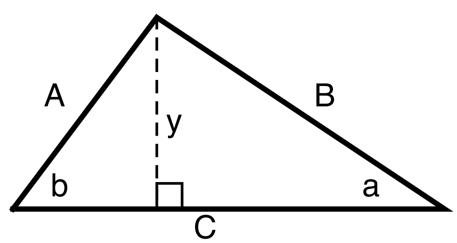
Now we notice that $x = B\cos\theta$ and $\theta = \pi - c$ so that we can say

$$x = -B\cos c$$

We finally substitute this into the expression earlier to get the Law of Cosines:

$$C^2 = A^2 + B^2 - 2AB\cos c$$

The Law of Sines



Given the triangle ABC shown above, we can drop an altitude to the base C with the dashed line *y* shown. Therefor we can say

$$y = A\sin b = B\sin a$$

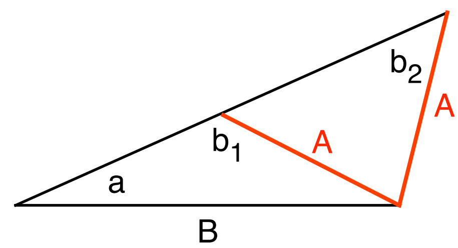
Which we can rewrite as:

$$\frac{\sin a}{A} = \frac{\sin b}{B}$$

Continue this idea by rotating the triangle and using either A or B as the base, drop a new altitude, and repeat above, except we will be using C and the non base side A or B. So we end up with the Law of Sines:

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

Pythagorean Theorem



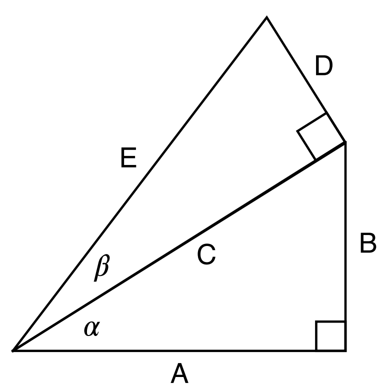
If you are using the Law of Sines with two angles and one side in order to find the other side, then there are no issues with the equation. However, you do have to be careful if you have one angle and two sides and are trying to find the missing angle. Sometimes (if A<B) there are two possible angles that can work, so you really need to pay attention.

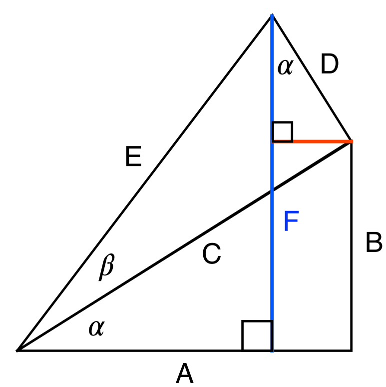
Look at the situation shown to the right. Imagine we are given the angle a, the side B and then a side A. Notice there are two possible triangles we can make, so there could be two possible angle b's, labeled b_1 and b_2. However, there is an isoscelese triangle with two sides of A in that diagram, so that means the angle next to b_1 is in fact b_2. Therefore the two possible angles, b_1 and b_2 are supplementary to each other. When using the Law of Sines, try and pay attention to the situation, and think about whether you need the larger or smaller answer in your problem.

Trig Identities

Since we are doing some trigonometry things, here are some trig identities with sine and cosine.

First, imagine a right triangle ABC, with angle α. Then make another right triangle CDE and angle β. This is shown in the diagram below left. Then drop an altitude F (shown in blue in the diagram below right) to make a right triangle with hypoteneuse E and height F. Lastly, draw a horizontal line (shown in red) to make the very small right triangle with hypoteneuse D. Notice in doing that, the "top" angle is also α.





From our first two right triangles we know that

$$\sin\alpha = \frac{B}{C} \qquad \cos\alpha = \frac{A}{C} \qquad \sin\beta = \frac{D}{E} \qquad \cos\beta = \frac{C}{E}$$

From the second diagram, we can also say

$$\sin(\alpha+\beta) = \frac{F}{E} \qquad\qquad F = B + D\cos\alpha$$

Subbing in for F and E we can say

$$\sin(\alpha+\beta) = \frac{B + D\cos\alpha}{C/\cos\beta}$$

Subbing in for D we can say

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$$\sin(\alpha+\beta) = \frac{C\sin\alpha + D\cos\alpha}{C/\cos\beta}$$

We can rewrite that as

$$\sin(\alpha+\beta) = \cos\beta\sin\alpha + \frac{D}{C}\cos\beta\cos\alpha$$

Subbing in for the second $\cos\beta$ we get

$$\sin(\alpha+\beta) = \cos\beta\sin\alpha + \frac{D}{C}\frac{C}{E}\cos\alpha$$

Finally, subbing in for D/E and rewriting a little we get the relationship

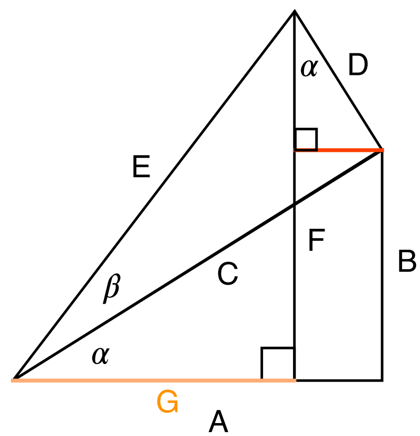
$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

The only time I can think of in which this identity is useful is actually the special case when the two angles are the same:

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

And since the triangles are already drawn, let's do the $\cos(\alpha+\beta)$ identity. In the diagram to the right, the base of our right triangle is now labeled G (so that the right triangle is GFE.) We have the same four relationships



$\sin\alpha = \frac{B}{C}$ $\cos\alpha = \frac{A}{C}$ $\sin\beta = \frac{D}{E}$ $\cos\beta = \frac{C}{E}$

Now we can also say

$\cos(\alpha+\beta) = \frac{G}{E}$ $G = A - D\sin\alpha$

Just as before, we will go through and make a bunch of substitutions based off the four original realtionships. So we have

$$\cos(\alpha+\beta) = \frac{A - D\sin\alpha}{C/\cos\beta}$$

$$\cos(\alpha+\beta) = \cos\beta\, {}^{A}/_{C} - {}^{D}/_{C}\cos\beta\sin\alpha$$

$$\cos(\alpha+\beta) = \cos\beta\cos\alpha - \tan\beta\cos\beta\sin\alpha$$

$$\cos(\alpha+\beta) = \cos\beta\cos\alpha - \sin\beta\sin\alpha$$

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Again, not sure that this will ever come up, but it's nice to have all these in the same place.

Pythagorean Theorem

Here is a slick way of deriving both the sine and cosine angle addition formulas, but you need to know Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$.

By Euler's Identity, we know that

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin\cos(\alpha+\beta).$$

Let's focus on the left half of that relationship as follows

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$e^{i(\alpha+\beta)} = \cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\sin\beta\cos\alpha - \sin\beta\sin\alpha$$

$$e^{i(\alpha+\beta)} = \cos\alpha\cos\beta - \sin\beta\sin\alpha + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$$

Equating this with our original statement we have

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

Since the real parts have to be equal and the imaginary parts have to be equal we have the two identities

$$\boxed{\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}}$$